

On the occupation times of Brownian excursions and Brownian loops

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Abstract We study properties of occupation times by Brownian excursions and Brownian loops in two-dimensional domains. This allows for instance to interpret some Gaussian fields, such as the Gaussian Free Fields as (properly normalized) fluctuations of the total occupation time of a Poisson cloud of Brownian excursions when the intensity of the cloud goes to infinity.

Keywords: Conformal invariance, Brownian excursion measure, Brownian loop measure, Green's function.

1 Introduction

Conformal invariance of planar Brownian motion has been derived and exploited long ago by Paul Lévy [8]. See also B. Davis (Annals of Proba 1979) in particular his derivation of Picard's big theorem. More recently, conformal invariance turned out to be an instrumental idea in the study of various critical models from statistical physics in the plane (see for instance [4, 16] and the references therein). Two basic important conformally invariant measures on random geometric objects are the Brownian excursion measure and the Brownian loop measure. Let us now very briefly describe these measures and the meaning of conformal invariance relatively to these measures. For each open domain D with non-polar boundary in the plane, one can define these two measures in D respectively denoted by μ_D and λ_D . These are infinite but σ -finite measures on Brownian-type paths with particular properties:

- μ_D is supported on the set of Brownian excursions $(B_t, t \leq \tau)$ in D i.e. Brownian paths such that B_0 and B_τ are in ∂D , while $B(0, \tau) \subset D$.

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- λ_D is supported on the set of Brownian loops $(B_t, t \leq \tau)$ i.e. Brownian paths in D such that $B_0 = B_\tau$.

In fact, in both cases, it is useful to view these paths up to monotone reparametrization (in the loop-case, one views the time-set modulo τ i.e., there is no “starting point” on the loop). Then, it turns out (see [5],[15] for details) that for any conformal map Φ from D onto $\Phi(D)$, the image measures of μ_D and λ_D under Φ are exactly $\mu_{\Phi(D)}$ and $\lambda_{\Phi(D)}$.

These two measures on loops and on excursions allow in some sense to get rid of the dependence of the measure on Brownian paths with respect to its starting point, see for instance the discussion in [16].

In the present text, we shall focus on the following type of results (here and in the sequel, dx or dy will denote the area measure, and x or y will always denote points in the plane):

Proposition 1. *Suppose that D is a simply connected domain and that A and B are two open proper subsets of D . Then,*

$$\mu_D\left(\int_0^\tau ds 1_A(\gamma_s) \int_0^\tau ds 1_B(\gamma_s)\right) = 4 \int_{A \times B} dx dy G_D(x, y) \quad (1)$$

and

$$\lambda_D\left(\int_0^\tau ds 1_A(\gamma_s) \int_0^\tau ds 1_B(\gamma_s)\right) = \int_{A \times B} dx dy (G_D(x, y))^2, \quad (2)$$

where $(\gamma_s, 0 \leq s \leq \tau)$ is a Brownian excursion in (1) and a Brownian loop in (2), $G_D(x, y)$ denotes the usual Green’s function in D (with Dirichlet boundary conditions).

The Brownian excursion measure and the loop measure are infinite measures, but they can be used to define random conformally invariant collections of excursions and loops (i.e. under a probability measure) by a Poissonization procedure. As explained in [16], both these Poissonian clouds are of interest and useful in the context of random planar conformally invariant curves of SLE-type: The “excursion clouds” give rise to the restriction measures [15], while the “loop-soups (loop clouds)” are related to Conformal Loop Ensembles (see [14]).

It is natural to study the cumulative occupation time of these random collections of Brownian paths. The previous proposition can then be viewed as a description of the covariance structure of these cumulative occupation times (even if as we shall explain later, things are slightly more complicated in the case of the loop measure because cumulative occupation times are infinite, so that a renormalization procedure is needed). By the classical central limit theorem, in the asymptotic regime where the intensity of these clouds goes to infinity, the fluctuations of these occupation times converge (if properly normalized of course) to a Gaussian process with the same covariance structure. This will in particular enable us to interpret the Gaussian Free Field in terms of fluctuations of occupation times of high-intensity clouds of Brownian excursions.

Note that in [7], a different and more direct (as it involves no asymptotic) relation between the loop-soup occupation times and the Gaussian Free Field (or rather its square) is pointed out.

Here is how the present paper is structured: In Section 2, we review various very elementary facts concerning Green's functions, their conformal invariance and their relation to Brownian motion and the Gaussian Free Field. In Section 3, we recall the definition of the Brownian excursion measure, we derive (1) and deduce from it the interpretation of the Gaussian Free Field as asymptotic fluctuations of the Excursions occupation time measure. In passing, we note a representation of the solution to the standard Dirichlet problem using Brownian excursions, that does not seem so well-known despite its simplicity. Section 4 is the counterpart of Section 3 for Brownian loops instead of Brownian excursions. Finally, in Section 5, we briefly mention a generalization of the previous results using some clouds of interacting pairs of excursions (via their intersection local-time) that exhibits some relations between loops and excursions.

We will focus on two-dimensional domains, but many of our statements (in particular those on Brownian excursions) are also valid in higher dimensions. However, as the reader will see, we choose to base our proofs on conformal invariance, so that another approach would be needed to derive the results in dimensions greater than two. We should also point out that the statements are in fact valid in non-simply connected domains, but again, some of our proofs, in particular those dealing with the loop-measure, would need to be changed in order to cover non-simply connected planar domains (as we will use explicit expressions for the unit disc).

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2 Review of basic notions

2.1 Generalities

We first recall some classical facts about Brownian motion and its relation to harmonic functions, see for instance [1, 10, 11] for further details or background.

Suppose that D is a bounded planar domain, and that it has a smooth boundary. Then, for any point x in D , the distribution of the exit position from D by a Brownian motion started at x has a continuous density with respect to the surface measure $\sigma(dz)$ on ∂D , called the *Poisson kernel*, that we will denote by $h_D(x, z)$ for $z \in \partial D$. In other words, the exit distribution is $h_D(x, z)\sigma(dz)$.

This Poisson kernel is closely related to the solutions of the *Dirichlet problem* in D (i.e., to find a harmonic function u in D , that is continuous on \bar{D} and equal to some prescribed continuous function f on the boundary of D). Indeed, the solution to the Dirichlet problem, if it exists, is given by

$$u(x) = \int \sigma(dz) h_D(x, z) f(z) = E_x(f(Z_\tau))$$

where Z is a planar Brownian motion started from x under the probability measure P_x and τ denotes its exit time from D .

The *Green's function* in D , is the unique function in $D \times D$, such that for each $x \in D$, $y \mapsto G_D(x, y)$ is harmonic, vanishes on ∂D , and satisfies $G_D(x, y) \sim \pi^{-1} \log(1/|x - y|)$ when $y \rightarrow x$.

Alternatively, one can think of $G_D(x, y)dy$ as the expected time spent by Z in the infinitesimal neighborhood of y before exiting D . More precisely, if A denotes an open set, the expected time spent by the Brownian motion Z (started from $Z_0 = x$) in A before exiting D is

$$E_x\left(\int_0^\tau dt 1_A(Z_t)\right) = \int_A dy G_D(x, y).$$

The Green's function is closely related to the *Poisson problem* (i.e. to find a C^2 function u in D such that $\Delta u = -2g$, where g is some given continuous function in D , with the property that u is continuous on \overline{D} and equal to 0 on ∂D). Under mild assumptions on D , the solution to this problem exists, is unique, and

$$u(x) = \int_D dy G_D(x, y) g(y) = E_x\left(\int_0^\tau dt f(Z_t)\right).$$

Not surprisingly, the Poisson kernel is closely related to the Green's function. More precisely, if $n = n_{z,D}$ is the inwards pointing normal vector at $z \in \partial D$, then, as ε goes to 0,

$$G_D(x, z + \varepsilon n) \sim 2\varepsilon h_D(x, z).$$

In the case of the unit disc $U := \{x : |x| < 1\}$ in the complex plane, the Poisson kernel and the Green's function can be explicitly computed:

$$h_U(x, z) = \frac{1 - |x|^2}{2\pi|x - z|^2}$$

and

$$G_U(x, y) = \frac{-1}{\pi} \log \frac{|x - y|}{|1 - x\bar{y}|}$$

for $x \in U, y \in U$, and $z \in \partial U$.

2.2 Conformal invariance

Conformal invariance of planar Brownian motion, first observed by Paul Lévy [8], can be described as follows: if one considers a planar Brownian motion Z started from x and stopped at its first exit time of a simply connected domain D , and if Φ

denotes a conformal map from D onto some other domain D' , then the law of $\Phi(Z)$ is that of a Brownian motion started from $\Phi(x)$ and stopped at its first exit time of D' . Actually, for this statement to be fully true, one has to reparametrize time of $\Phi(Z)$ in a proper way. The rigorous statement is that for all $t < \tau$,

$$\Phi(Z_t) = Z'_{H_t} \text{ with } H_t = \int_0^t ds |\Phi'(Z_s)|^2,$$

where Z' is a Brownian motion started from $\Phi(x)$, stopped at $\tau' = H_\tau$, which is its exit time of D' .

Conformal invariance of Brownian motion is closely related to the conformal invariance of the Green's function and of the Poisson kernel. Let us give a rather convoluted explanation of the conformal invariance of Green's functions using Brownian motion (a direct proof using the analytic characterization of the Green's function is much more straightforward) that will be helpful for what follows. Suppose that x and y are in D and that ε is very small. We have seen that the expected time spent in the ball $U(y, \varepsilon)$, centered at y and of radius ε , by the Brownian motion Z started at x behaves like

$$\pi \varepsilon^2 G_D(x, y)$$

when $\varepsilon \rightarrow 0$. Equivalently, the expected time spent in the ball $U(\Phi(y), |\Phi'(y)|\varepsilon)$ by the Brownian motion β started at $\Phi(x)$, behaves like

$$\pi |\Phi'(y)|^2 \varepsilon^2 G_{D'}(\Phi(x), \Phi(y))$$

as $\varepsilon \rightarrow 0$. The process $\Phi(Z)$ can be viewed as a time-changed Brownian motion, and the time-change when Z is close to y is described via H_t . It follows easily that this expected time of $\Phi(Z)$ spent in the ball $U(\Phi(y), |\Phi'(y)|\varepsilon)$ behaves like

$$\frac{\pi |\Phi'(y)|^2 \varepsilon^2 G_{D'}(\Phi(x), \Phi(y))}{|\Phi'(y)|^2} = \pi \varepsilon^2 G_D(x, y).$$

As a result, we have indeed that

$$G_{\Phi(D)}(\Phi(x), \Phi(y)) = G_D(x, y). \quad (3)$$

For a more rigorous derivation along the same lines, we can use the integral representation of occupation times of domains : on the one hand,

$$\begin{aligned} E_{\Phi(x)}\left(\int_0^{\tau_{D'}} dt f(Z'_t)\right) &= \int_{D'} dy G_{D'}(\Phi(x), y) f(y) \\ &= \int_D |\Phi'(y)|^2 dy G_{D'}(\Phi(x), \Phi(y)) f(\Phi(y)) \end{aligned}$$

for indicator functions $f = 1_A$, and on the other hand,

$$\begin{aligned}
E_{\Phi(x)}\left(\int_0^{\tau_{D'}} dt f(Z'_t)\right) &= E_x\left(\int_0^{\tau_D} |\Phi'(Z_t)|^2 dt f(\Phi(Z_t))\right) \\
&= \int_D dy G_D(x, y) f(\Phi(y)) |\Phi'(y)|^2.
\end{aligned}$$

Conformal invariance of planar Brownian motion can also be used in a similar way to see that

$$|\Phi'(z)| h_{\Phi(D)}(\Phi(x), \Phi(z)) = h_D(x, z) \quad (4)$$

for all $x \in D, z \in \partial D$ when ∂D is smooth. Let us stress again that these conformal invariance properties of the Green's functions and of the Poisson kernel can be derived much more directly without any reference to Brownian paths.

Note that $G_U(0, y_0) = -\pi^{-1} \log |y_0|$ for all $y_0 \neq 0$. The formula for $G_U(x, y)$ then follows immediately, using the Möbius transformation ϕ_x of U onto itself that maps x onto 0 and vice-versa (this is the map $z \mapsto (z - x)/(1 - \bar{x}z)$) because then $G_U(x, y) = G_U(0, \phi_x(y))$. Note also that this conformal invariance also provides one possible explanation of the symmetry of the Green's function $G_U(x, y) = G_U(y, x)$ (because for any x and y , there exists a conformal map from D onto itself that maps x onto y and y onto x).

Similarly, since clearly $h_U(0, z) = 1/(2\pi)$ for all $z \in \partial U$, the formula for $h_U(x, z)$ recalled at the end of the previous subsection follows using conformal invariance.

2.3 The Gaussian Free Field

In the present text, we will briefly relate our Brownian excursions to the Gaussian Free Field, which is a classical and basic building block in Field theory, see for instance [9, 2]. So we recall its definition, in the Gaussian Hilbert space framework (as in [12] for instance): Consider the space $H_s(D)$ of smooth, real-valued functions on R^2 that are supported on a compact subset of a domain $D \subset R^d$ (so that, in particular, their first derivatives are in $L^2(D)$). This space can be endowed with the *Dirichlet inner product* defined by

$$(f_1, f_2)_\nabla = \int_D dx (\nabla f_1 \cdot \nabla f_2)$$

It is immediate to see that this Dirichlet inner product is invariant under conformal transformation. Denote by $H(D)$ the Hilbert space completion of $H_s(D)$. The quantity $(f, f)_\nabla$ is called the *Dirichlet energy* of f .

A *Gaussian Free Field* is any Gaussian Hilbert space $\mathcal{G}(D)$ of random variables denoted by “ $(h, f)_\nabla$ ”—one variable for each $f \in H(D)$ —that inherits the Dirichlet inner product structure of $H(D)$, i.e.,

$$E[(h, a)_\nabla (h, b)_\nabla] = (a, b)_\nabla.$$

In other words, the map from f to the random variable $(h, f)_\nabla$ is an inner product preserving map from $H(D)$ to $\mathcal{G}(D)$. The reason for this notation is that it is possible to view h as a random linear operator, but we will not need this approach. We also view (h, ρ) as being well defined for all $\rho \in (-\Delta)H(D)$ (if $\rho = -\Delta f$ for some $f \in H(D)$), then we denote $(h, \rho) = (h, f)_\nabla$.

When ρ_1 and ρ_2 are in $H_s(D)$, the covariance of (h, ρ_1) and (h, ρ_2) can be written as $(-\Delta^{-1}\rho_1, -\Delta^{-1}\rho_2)_\nabla = (\rho_1, -\Delta^{-1}\rho_2) = (-\Delta^{-1}\rho_1, \rho_2)$. From the Poisson problem that we discussed before, $-\Delta^{-1}\rho$ can be written using the Green's function as

$$[-\Delta^{-1}\rho](x) = \frac{1}{2} \int_D dy G_D(x, y) \rho(y),$$

we may also write:

$$\text{Cov}[(h, \rho_1), (h, \rho_2)] = \frac{1}{2} \int dx dy G_D(x, y) \rho_1(x) \rho_2(y) \quad (5)$$

Both the Dirichlet inner product and the Gaussian Free Field inherit naturally conformal invariance properties from the conformal invariance of the Green's function. The 2-dimensional Gaussian free field (GFF) is a particular rich object, in which a number of geometric features can be detected, and that has been shown to play a central role in the theory of random surfaces and conformally invariant geometric structures, see [13] and the references therein.

3 Occupation times of Brownian excursions

Brownian excursion measure. Let us first very briefly recall the construction of Brownian excursion measures. For the unit disc U , for each $\varepsilon > 0$, let μ_ε denote the measure of total mass $1/\varepsilon$ defined as $1/\varepsilon$ times the law of a Brownian motion started uniformly on the circle of radius $(1 - \varepsilon)$, and stopped at its first hitting time of the unit circle. In some appropriate topology, the measures μ_ε converge when $\varepsilon \rightarrow 0$ to an infinite measure μ on two-dimensional paths that start and end on the unit circle. For a general simply connected domain D , the excursion measure μ_D can either be defined as the image of μ by the conformal map Φ that maps U onto D , or alternatively in an analogous way as in the disc, by integrating over the choice of the starting point of the excursion on ∂D . The fact that these two definitions are equivalent is the conformal invariance property of the Brownian excursion measures. See e.g. [16] for details and references.

Note that μ is a measure on paths $(B_t, 0 < t < \tau)$ that start and end on ∂D (i.e., $B_0 \in \partial D$ and $B_\tau \in \partial D$) that are “oriented”, i.e. B_0 and B_τ do a priori not play the same role. However, it turns out that the Brownian excursions are reversible i.e., that $(B_t, 0 < t < \tau)$ and $(B_{\tau-t}, 0 < t < \tau)$ are defined under the same measure (this can for instance be easily seen using the definition in the case where D is the upper half-plane).

Brownian excursion occupation times and the Dirichlet problem. Let us first make a comment on the relation between the Brownian excursion measure and the Dirichlet problem. Let u be the solution to the Dirichlet problem, i.e. $\Delta u = 0$ in U and $u = f$ on ∂U . For all $z \in \partial U$ and all positive ε , we have that

$$\begin{aligned}
E_{(1-\varepsilon)z}(\int_0^\tau dt 1_A(\gamma_t) f(\gamma_\tau)) &= E_{(1-\varepsilon)z}(\int_0^\infty dt 1_A(\gamma_t) 1_{t \leq \tau} f(\gamma_\tau)) \\
&= E_{(1-\varepsilon)z}(\int_0^\infty dt 1_A(\gamma_t) 1_{t \leq \tau} E(f(\gamma_\tau) | \mathcal{F}_t)) \\
&= E_{(1-\varepsilon)z}(\int_0^\tau dt 1_A(\gamma_t) E_{\gamma_t}(f(\gamma_\tau))) \\
&= E_{(1-\varepsilon)z}(\int_0^\tau dt 1_A(\gamma_t) u(\gamma_t)) \\
&= \int_A dy G_U((1-\varepsilon)z, y) u(y)
\end{aligned}$$

And for the Brownian excursion measure $\mu = \mu_U$, we have that

$$\begin{aligned}
\mu(\int_0^\tau dt 1_A(\gamma_t) f(\gamma_\tau)) &= \lim_{\varepsilon \rightarrow 0} \int_0^{2\pi} \frac{d\theta}{\varepsilon} E_{(1-\varepsilon)e^{i\theta}}(\int_0^\tau dt 1_A(\gamma_t) f(\gamma_\tau)) \\
&= \lim_{\varepsilon \rightarrow 0} \int_0^{2\pi} \frac{d\theta}{\varepsilon} \int_A dy G_U((1-\varepsilon)e^{i\theta}, y) u(y) \\
&= \int_0^{2\pi} 2d\theta \int_A dy h_U(y, e^{i\theta}) u(y) \\
&= 2 \int_A dy u(y) \int_0^{2\pi} d\theta h_U(y, e^{i\theta}) \\
&= 2 \int_A dy u(y)
\end{aligned}$$

That is to say, we can represent the solution to the Dirichlet problem via the Brownian excursion measure by the formula

$$\mu(\int_0^\tau dt 1_A(\gamma_t) f(\gamma_\tau)) = 2 \int_A dy u(y)$$

Since the Brownian excursion is reversible, we also have that

$$\mu(f(\gamma_0) \int_0^\tau dt 1_A(\gamma_t)) = 2 \int_A dy u(y) \quad (6)$$

Hence, if we put a weight f on starting point of the excursion, then the mean occupation time spent in A is measured by the integral of u on A , where u is the solution to the corresponding Dirichlet problem. By conformal invariance, (6) also holds for any simply connected domain.

We would like to note that, if we set $f = 1$ in (6), we get that $\mu_D(\int_0^\tau dt 1_A(\gamma_t))$ is equal to twice the area of A . In particular, $\mu_D(\tau)$ is therefore just twice the area of D .

The covariance structure. We now turn our attention towards the proof of (1). This formula can be understood as follows: we can cut $A \times B$ into very small pieces, calculate on each small piece and then add all these pieces together. On each small piece $dx \times dy$, the Brownian excursion starts from the boundary, firstly it hits the small piece dx (with a small probability), after this time, it is a true Brownian motion starting nearby x , which is (almost) independent of the past and then the expected time of this new Brownian motion spent in the neighborhood of y before exiting D is close to $G_D(x, y)dy$. When we add up all these small pieces together and we obtain the right-hand side of the formula.

For a precise calculation, we first consider the case where $D = U$ as the general case will then follow from conformal invariance. We also use the notation that $\mu = \mu_U$. Let γ denote a Brownian excursion in U . For all $z \in \partial U$ and all positive ε ,

$$\begin{aligned} & E_{(1-\varepsilon)z} \left(\int_0^\tau ds 1_A(\gamma_s) \int_s^\tau dt 1_B(\gamma_t) \right) \\ &= E_{(1-\varepsilon)z} \left(\int_0^\tau ds 1_A(\gamma_s) E \left(\int_s^\tau dt 1_B(\gamma_t) \middle| \mathcal{F}_s \right) \right) \\ &= E_{(1-\varepsilon)z} \left(\int_0^\tau ds 1_A(\gamma_s) E_{\gamma_s} \left(\int_0^\tau dt 1_B(\gamma_t) \right) \right) \\ &= E_{(1-\varepsilon)z} \left(\int_0^\tau ds 1_A(\gamma_s) G_U(\gamma_s, B) \right) \\ &= \int_A dy G_U((1-\varepsilon)z, y) G_U(y, B). \end{aligned}$$

And for the Brownian excursion measure, we have that

$$\begin{aligned} & \mu \left(\int_0^\tau ds 1_A(\gamma_s) \int_s^\tau dt 1_B(\gamma_t) \right) \\ &= \lim_{\varepsilon \rightarrow 0} \int_0^{2\pi} \frac{d\theta}{\varepsilon} E_{(1-\varepsilon)e^{i\theta}} \left(\int_0^\tau ds 1_A(\gamma_s) \int_s^\tau dt 1_B(\gamma_t) \right) \\ &= \lim_{\varepsilon \rightarrow 0} \int_0^{2\pi} \frac{d\theta}{\varepsilon} \int_A dy G_U((1-\varepsilon)e^{i\theta}, y) G_U(y, B) \\ &= \int_0^{2\pi} 2d\theta \int_A dy h_U(y, e^{i\theta}) G_U(y, B) \\ &= 2 \int_A dy G_U(y, B) \int_0^{2\pi} d\theta h_U(y, e^{i\theta}) \\ &= 2 \int_A dy G_U(y, B) \end{aligned}$$

By symmetry of the Green's function ($G_U(x, y) = G_U(y, x)$), we have that

$$\mu \left(\int_0^\tau ds 1_A(\gamma_s) \int_0^\tau ds 1_B(\gamma_s) \right) = 4 \int_{A \times B} dx dy G_U(x, y).$$

This concludes the proof of the equation (1), since we can use to conformal invariance to derive the formula for general simply connected domain D . More generally,

we have that

$$\mu_D(\int_0^\tau ds f(\gamma_s) \int_0^\tau ds g(\gamma_s)) = 4 \int dx dy G_D(x, y) f(x) g(y) \quad (7)$$

for all measurable bounded functions f and g .

Large intensity clouds of excursions and GFF. Let us now use this formula to make a link between Brownian excursions and the GFF. For this we are going to use Poissonian cloud of excursions in D , as in [15]. Recall that a Poisson cloud of excursions with intensity $c\mu_D$ is a random countable family of Brownian excursions in D , that is defined as a Poisson point process with intensity $c\mu_D$.

In particular, the union of two independent Poissonian clouds of Brownian excursions in D with intensity $c_1\mu_D$ and $c_2\mu_D$ is a Poissonian cloud of excursions in D with intensity $(c_1 + c_2)\mu_D$.

Let us now consider an i.i.d. sequence $M^j, j \geq 1$ of Poissonian clouds of excursions in D with the common intensity μ_D . For each $j \geq 1$, and each $f \in (-\Delta)H(D)$, define the “cumulative occupation” time of M^j by

$$X_f^j = \sum_{\gamma \in M^j} \int_0^{\tau(\gamma)} ds f(\gamma_s).$$

The fact that $\mu(\tau)$ is finite (as soon as the area of D is finite) ensures that X_f^j is almost surely finite (as soon as f is bounded) because its expectation is bounded. We then define

$$\tilde{X}_f^j = X_f^j - E(X_f^j).$$

On an enlarged probability space, we can also define an i.i.d. family of random variable ε_γ indexed by the set of excursions in $\cup_j M^j$ such that $P(\varepsilon_\gamma = 1) = P(\varepsilon_\gamma = -1) = 1/2$. We can then define

$$Y_f^j = \sum_{\gamma \in M^j} \varepsilon_\gamma \int_0^{\tau(\gamma)} ds f(\gamma_s).$$

It is easy to see that $Y_f^1, Y_f^2, Y_f^3, \dots$ are i.i.d. centered random variables with common variance

$$\sigma_f^2 = \mu_D(\int_0^\tau ds f(\gamma_s) \int_0^\tau ds f(\gamma_s)) = 4 \int dx dy G_D(x, y) f(x) f(y).$$

The same is true for $\tilde{X}_f^1, \tilde{X}_f^2, \tilde{X}_f^3, \dots$. By the Central Limit Theorem, we have that

$$\frac{1}{\sqrt{N}}(Y_f^1 + \dots + Y_f^N)$$

converges in law as $N \rightarrow \infty$ to a centered Gaussian random variable Y_f with variance σ_f^2 . The same holds for the sequence $(\tilde{X}_f^1 + \dots + \tilde{X}_f^N)/\sqrt{N}$.

Hence, we see that the GFF can be viewed as the limit (in law, and in the sense of finite-dimensional distributions) of the occupation times fluctuations of a Poisson cloud of Brownian excursions, when the intensity tends to infinity.

Higher-order “moments”. We just mention that our proof can be adapted directly in order to show that for all $p \geq 2$:

$$\begin{aligned} \mu_D\left(\int_{(0,\tau)^p} dt_1 \dots dt_p 1_{t_1 < \dots < t_p} 1_{A_1}(\gamma_1) \dots 1_{A_p}(\gamma_p)\right) \\ = 2 \int_{A_1 \times \dots \times A_p} dx_1 \dots dx_p G_D(x_1, x_2) \times \dots \times G_D(x_{p-1}, x_p) \end{aligned}$$

which gives for instance (when one sums over all possible order of visits) a formula for $\mu_D((\int_0^\tau f(\gamma_s) ds)^p)$. We have chosen to focus on the case $p = 2$ because of the above-mentioned link with Gaussian fields.

Non-simply-connected domains. Suppose that D is a finitely connected open domain in the plane. Then, by Koebe’s uniformization Theorem (see [3]), it is possible to map it conformally onto a circular domain i.e., the unit disk U punctured by a finite number of disjoint closed disks. It is very easy to generalize the definition of the Brownian excursion measure in circular domains (adding the contributions corresponding to starting points in the neighborhood of each of the boundary disks), and to see that all our proofs go through without any real difficulty, so that all our statements are in fact valid also in circular domains. One can then *define* the excursion measure in D via conformal invariance starting from circular domains, and then, by conformal invariance of all the quantities involved, we easily see that all our statements are also valid in D .

4 Occupation times of Brownian loops

Brownian loop measure. We now briefly recall the construction of the Brownian loop measure [5]. As for the Brownian excursion measure, we can first define it in the unit disc, and then define it in any other simply connected domain using conformal invariance (and one then checks that this is indeed consistent with other possible constructions).

For any $r \in (0, 1]$, define $U_r = rU$. For any $x \in U_r$ and any $z \in \partial U_r$, one can define the Brownian motion started at x and conditioned to exit U_r at z (this can be rigorously defined as the limit when $\varepsilon \rightarrow 0$ of the law of the Brownian motion conditioned to exit U_r in an ε -neighborhood of z). Let us denote this probability measure by $P_{x \rightarrow z}^r$. Then, as for the excursion measure, one can let $x \rightarrow z$, and renormalize it in order to get a measure on macroscopic sets i.e. define

$$m_z^r(\cdot) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} h_{U_r}(z + \varepsilon n, z) P_{z + \varepsilon n \rightarrow z}^r(\cdot)$$

where $n = n_{z, U_r}$ is the inwards pointing normal vector at $z \in \partial U_r$. Then, one can define the loop measure in U by integrating z on ∂U_r , and then integrating r from 0 to 1:

$$\lambda_U(\cdot) = \int_0^1 r dr \int_0^{2\pi} d\theta m_{re^{i\theta}}^r(\cdot).$$

In fact, the above definition is not quite the loop measure because it defines a measure on parametrized loops. We will forget about the precise parametrization of the loop and view λ_U as a measure on loops defined modulo monotone reparametrization (where the time-parameter should be viewed as an element of the circle, because the end-point of the loop is the same as the starting point, this is possible). It turns out that this definition of λ_U is then invariant under the Moebius transformations that map the unit disc onto itself. Hence, it is possible to define, for a general simply connected domain D , the loop measure λ_D as the image of λ_U by any conformal map Φ that maps U onto D . And we usually denote $\lambda = \lambda_U$.

Before going on, we would like to say a word on the value of $\lambda(\tau)$. In fact, by direct computation we have that $\lambda(\tau) = \infty$ which is very different from $\mu(\tau)$ mentioned before. A direct way to check that $\lambda(\tau) = \infty$ goes as follows. Consider D to be the square $[0, 1]^2$. For any dyadic square d in D with sidelength 2^{-n} , a direct scaling argument shows that the mass (for λ) of the set of loops that stay in d and have a time-length in $[4^{-n}, 2 \times 4^{-n})$ does not depend on d . Hence, if we sum this quantity over all dyadic squares d in D , and because $\sum_n 4^n 4^{-n} = \infty$, we readily see that $\lambda(\tau) = \infty$.

However, almost the same argument ensures that $\lambda(\tau^{1+\varepsilon})$ is finite for $\varepsilon > 0$ (and bounded D). Indeed, in the case of the unit square, we can decompose the set of loops with time-length in $[4^{-n}, 4^{1-n})$ according to the dyadic square in which its lowest point lies. This leads readily to the bound

$$\lambda(1_{\tau < 1} \tau^{1+\varepsilon}) \leq C \sum_{n \geq 1} 4^n (4^{1-n})^{1+\varepsilon} < \infty$$

and one can see by other means that $\lambda(\tau > t)$ decays exponentially fast as $t \rightarrow \infty$. In particular, we get that $\lambda(\tau^2)$ is finite (as soon as D is bounded).

Covariance structure. Our goal is now to prove (2). As before, we are going to derive the result first in the case where $D = U$, and the general result will then follow using conformal invariance. Again, it will be convenient to (loosely speaking) divide $A \times B$ into infinitesimal pieces $dx \times dy$, make the computation on each piece, and then add all these pieces together. Clearly, this will give a formula of the type

$$\lambda_D\left(\int_0^\tau ds 1_A(\gamma_s) \int_0^\tau ds 1_B(\gamma_s)\right) = \int_{A \times B} dx dy F_D(x, y)$$

where $F_D(x, y)$ is the ‘‘covariance’’ function between x and y determined by the Brownian loop measure. Just as what we have done to derive the conformal invariance of the Green’s function in the equation (3), we can also derive the conformal invariance of F :

$$F_{\Phi(D)}(\Phi(x), \Phi(y)) = F_D(x, y).$$

To determine $F_D(x, y)$, it is enough to describe $F_U(0, y_0)$ for $y_0 \in (0, 1)$, because there exists a y_0 and a conformal map Φ from D onto U such that $\Phi(x) = 0$, $\Phi(y) = y_0$.

And now begin our computation. For $r \in (0, 1)$, $z \in \partial U_r$, we can write

$$\begin{aligned}
& E_{z+\varepsilon n \rightarrow z}^r \left(\int_0^\tau ds 1_A(\gamma_s) \int_s^\tau dt 1_B(\gamma_t) \right) \\
&= \lim_{\varepsilon' \rightarrow 0} \left(\frac{1}{P_{z+\varepsilon n}^r(\gamma_\tau \in U(z, \varepsilon') \cap \partial U_r)} \right. \\
&\quad \left. \times E_{z+\varepsilon n}^r \left(\int_0^\tau ds 1_A(\gamma_s) \int_s^\tau dt 1_B(\gamma_t) 1_{\gamma_\tau \in U(z, \varepsilon') \cap \partial U_r} \right) \right) \\
&= \lim_{\varepsilon' \rightarrow 0} \left(\frac{1}{h_{U_r}(z + \varepsilon n, U(z, \varepsilon') \cap \partial U_r)} \right. \\
&\quad \left. \times E_{z+\varepsilon n}^r \left(\int_0^\tau ds 1_A(\gamma_s) \int_s^\tau dt 1_B(\gamma_t) h_{U_r}(\gamma_t, U(z, \varepsilon') \cap \partial U_r) \right) \right) \\
&= \frac{1}{h_{U_r}(z + \varepsilon n, z)} E_{z+\varepsilon n}^r \left(\int_0^\tau ds 1_A(\gamma_s) \int_s^\tau dt 1_B(\gamma_t) h_{U_r}(\gamma_t, z) \right).
\end{aligned}$$

Hence,

$$\begin{aligned}
& h_{U_r}(z + \varepsilon n, z) E_{z+\varepsilon n \rightarrow z}^r \left(\int_0^\tau ds 1_A(\gamma_s) \int_s^\tau dt 1_B(\gamma_t) \right) \\
&= E_{z+\varepsilon n}^r \left(\int_0^\tau ds 1_A(\gamma_s) \int_s^\tau dt 1_B(\gamma_t) h_{U_r}(\gamma_t, z) \right) \\
&= \int_A dx G_{U_r}(z + \varepsilon n, x) \int_B dy G_{U_r}(x, y) h_{U_r}(y, z)
\end{aligned}$$

and letting $\varepsilon \rightarrow 0$, we get

$$\begin{aligned}
& m_z^r \left(\int_0^\tau ds 1_A(\gamma_s) \int_0^\tau ds 1_B(\gamma_s) \right) \\
&= \lim_{\varepsilon \rightarrow 0} \frac{2}{\varepsilon} \int_A dx G_{U_r}(z + \varepsilon n, x) \int_B dy G_{U_r}(x, y) h_{U_r}(y, z) \\
&= 4 \int_{A \times B} dx dy G_{U_r}(x, y) h_{U_r}(x, z) h_{U_r}(y, z).
\end{aligned}$$

For simplicity, we define a new kernel

$$K_{U_r}(x, y) = 4 \int_0^{2\pi} d\theta h_{U_r}(x, re^{i\theta}) h_{U_r}(y, re^{i\theta})$$

and then we have that

$$\lambda_U \left(\int_0^\tau ds 1_A(\gamma_s) \int_0^\tau ds 1_B(\gamma_s) \right) = \int_0^1 r dr \int_{A \times B} dx dy G_{U_r}(x, y) K_{U_r}(x, y).$$

Note that

$$K_{U_r}(rx, ry) = \frac{1}{r^2} K_U(x, y)$$

and

$$G_{U_r}(rx, ry) = G_U(x, y).$$

Furthermore, $K_U(0, y) = 2/\pi$.

Suppose that $A = U(0, \varepsilon)$ and $B = U(y_0, \delta)$ where ε and δ are both small. In our decomposition of λ_U , the loop can visit B only if it started on a circle of radius $r > y_0$. Hence, on the one hand, as ε and δ tend to 0,

$$\begin{aligned} \lambda_U\left(\int_0^\tau ds 1_A(\gamma_s) \int_0^\tau ds 1_B(\gamma_s)\right) &= \int_{y_0}^1 r dr \int_{(A \cap U_r) \times (B \cap U_r)} dx dy G_{U_r}(x, y) K_{U_r}(x, y) \\ &\sim \int_{y_0}^1 r dr (\pi \varepsilon^2 \pi \delta^2) G_{U_r}(0, y_0) K_{U_r}(0, y_0) \\ &= (\pi \varepsilon^2 \pi \delta^2) \int_{y_0}^1 \frac{1}{r} dr G_U(0, \frac{y_0}{r}) K_U(0, \frac{y_0}{r}) \\ &= (\pi \varepsilon^2 \pi \delta^2) \frac{2}{\pi^2} \int_{y_0}^1 dr \left(-\frac{1}{r} \log\left(\frac{y_0}{r}\right)\right) \\ &= (\pi \varepsilon^2 \pi \delta^2) \frac{1}{\pi^2} (\log y_0)^2 \end{aligned}$$

On the other hand, this quantity is precisely behaving as $(\pi \varepsilon^2 \pi \delta^2) F_U(0, y_0)$ and as a result, we get that

$$F_U(0, y_0) = \frac{1}{\pi^2} (\log y_0)^2 = (G_U(0, y_0))^2.$$

We can then conclude that (2) holds in U , and then also in D by conformal invariance. More generally, we have that

$$\lambda_D\left(\int_0^\tau ds f(\gamma_s) \int_0^\tau ds g(\gamma_s)\right) = \int_{A \times B} dx dy (G_D(x, y))^2 f(x) g(y).$$

for all measurable bounded functions f and g .

Brownian loop-soups and fields. Just as in the case of Brownian excursion measure, we can use this formula to make a link between Brownian loops and some Gaussian Fields. Let $M^j, j \geq 1$ be a sequence of i.i.d Poissonian clouds of loops in D with the common intensity λ_D . We can try to give the same definitions of the quantities $\tilde{X}_f^j, Y_f^j, j \geq 1$. However, things are a little more complicated, due to the fact that the same scaling argument that showed that $\lambda(\tau) = \infty$ implies that

$$\sum_{\gamma \in M^j} \tau(\gamma) = \infty$$

almost surely, so that some care is needed.

The definition of Y_f^j is however not a big problem. Recall that on an enlarged probability space, one associates to each loop γ a random variable ε_γ with $E(\varepsilon_\gamma) = 0$ and $E((\varepsilon_\gamma)^2) = 1$. But equation (2) precisely ensures that the sum

$$\sum_{\gamma \in M^j} \varepsilon_\gamma \int_0^{\tau(\gamma)} f(\gamma_s) ds$$

makes sense in L^2 , and that its second moment is equal to

$$\sigma_f^2 = \lambda_D \left(\int_0^\tau ds f(\gamma_s) \int_0^\tau ds f(\gamma_s) \right) = \int dx dy (G_D(x, y))^2 f(x) f(y)$$

which is finite.

Then, just as in the case of the clouds of excursions, the sequence Y_f^1, Y_f^2, \dots is made of i.i.d centered random variables with common variance σ_f^2 . By the Central Limit Theorem,

$$\frac{1}{\sqrt{N}} (Y_f^1 + \dots + Y_f^N)$$

converges in law as $N \rightarrow \infty$ to a centered Gaussian random variable with variance σ_f^2 . Hence, we obtain another Gaussian Field, characterized by this new covariance structure.

It is also still possible to make sense of \tilde{X}_f^j even though it is not possible to define X_f^j . It suffices to partition the set of loops (in D) into a countable set of loops $A_k, k \geq 1$ such that for each k , $\lambda(\tau 1_{\gamma \in A_k})$ is finite (for instance, one can take $A_k = \{\gamma : \tau(\gamma) > 1/k\} \setminus (A_1 \cup \dots \cup A_{k-1})$). Then, one can define

$$\tilde{X}_f^j = \sum_{k \geq 1} \left(\sum_{\gamma \in A_k \cap M^j} \int_0^{\tau(\gamma)} f(\gamma_s) ds - E \left(\sum_{\gamma \in A_k \cap M^j} \int_0^{\tau(\gamma)} f(\gamma_s) ds \right) \right)$$

and check that this sum with respect to k converges in L^2 , and that its second moment is the same as that of Y_f . The rest of the argument is again the same.

5 Intersections of Brownian excursions

In this section, we try to find the relation between intersections of Brownian excursion “occupations times” and Brownian loop occupation times, the former being defined via the intersection local time.

Let us first recall some features of Brownian intersection local times. Let $p \geq 2$ be an integer, and let Z^1, \dots, Z^p denote p independent Brownian motions in R^2 , started at x^1, \dots, x^p respectively. The intersection local time of Z^1, \dots, Z^p is a random measure $\alpha(ds_1 \dots ds_p)$ on R_+^p , supported on

$$\{(s_1, \dots, s_p) \in R_+^p : Z_{s_1}^1 = \dots = Z_{s_p}^p\}.$$

The basic description concerning the intersection local time that we will use goes as follows (see [6] for details):

Proposition 2. *Almost surely, one can define a (random) measure $\alpha(ds_1 \dots ds_p)$ on R_+^p such that, for any A^1, \dots, A^p bounded Borel subsets of R_+ ,*

$$\alpha(A^1 \times \dots \times A^p) = \lim_{\varepsilon \rightarrow 0} \alpha_\varepsilon(A^1 \times \dots \times A^p)$$

in the L^n -norm, for any $n < \infty$, where

$$\alpha_\varepsilon(ds_1 \dots ds_p) = ds_1 \dots ds_p \int_{R^2} dy \delta_y^\varepsilon(Z_{s_1}^1) \dots \delta_y^\varepsilon(Z_{s_p}^p)$$

with $\delta_y^\varepsilon(z) = \frac{1}{\pi \varepsilon^2} 1_{U(y, \varepsilon)}(z)$.

Let us use this in the context of the Brownian excursion measure. This time we shall consider two Brownian excursions γ and γ' defined under the (infinite) measure $\mu_D \otimes \mu_D$, and study the behavior of their intersection local time that spent in two disjoint sets A and B , as before:

$$\begin{aligned} & \mu_D \otimes \mu_D \left(\int_0^\tau \int_0^{\tau'} \alpha(dt dt') 1_{(\gamma = \gamma' \in A)} \int_0^\tau \int_0^{\tau'} \alpha(ds ds') 1_{(\gamma = \gamma' \in B)} \right) \\ &= \lim_{\varepsilon \rightarrow 0} \mu_D \otimes \mu_D \left(\int_0^\tau \int_0^{\tau'} \alpha_\varepsilon(dt dt') 1_{(\gamma \in A)} 1_{(\gamma' \in A)} \right. \\ & \quad \left. \int_0^\tau \int_0^{\tau'} \alpha_\varepsilon(ds ds') 1_{(\gamma \in B)} 1_{(\gamma' \in B)} \right) \\ &= \lim_{\varepsilon \rightarrow 0} \mu_D \otimes \mu_D \left(\int_0^\tau \int_0^{\tau'} dt dt' \int dx \delta_x^\varepsilon(\gamma) \delta_x^\varepsilon(\gamma') 1_{(\gamma \in A)} 1_{(\gamma' \in A)} \right. \\ & \quad \left. \int_0^\tau \int_0^{\tau'} ds ds' \int dy \delta_y^\varepsilon(\gamma) \delta_y^\varepsilon(\gamma') 1_{(\gamma \in B)} 1_{(\gamma' \in B)} \right) \\ &= \lim_{\varepsilon \rightarrow 0} \int dx \int dy \mu_D \otimes \mu_D \left(\int_0^\tau dt \delta_x^\varepsilon(\gamma) 1_{(\gamma \in A)} \int_0^\tau ds \delta_y^\varepsilon(\gamma) 1_{(\gamma \in B)} \right. \\ & \quad \left. \int_0^{\tau'} dt' \delta_x^\varepsilon(\gamma') 1_{(\gamma' \in A)} \int_0^{\tau'} ds' \delta_y^\varepsilon(\gamma') 1_{(\gamma' \in B)} \right) \\ &= \lim_{\varepsilon \rightarrow 0} \int dx \int dy \left(4 \int_{A \times B} da db \delta_x^\varepsilon(a) \delta_y^\varepsilon(b) G_D(a, b) \right)^2 \\ &= 16 \int_{A \times B} dx dy (G_D(x, y))^2 \end{aligned}$$

Hence, we see that pairs of Brownian excursions give rise to the same covariance structure as the Brownian loops. In a way, this is not too surprising, as for two points x and y that are both visited by γ and by γ' , one sees in a way a loop structure (the part of γ from x to y , and then the part of γ' back from y to x).

Note that by a similar calculation, one gets that for any $p \geq 3$, if one defines for any A ,

$$T_p(A; \gamma^1, \dots, \gamma^p) = \int_0^{\tau_1} \dots \int_0^{\tau_p} \alpha(dt_1 \dots dt_p) 1_{(\gamma_1^1 = \dots = \gamma_p^p \in A)},$$

then

$$\mu_D^{\otimes p}(T_p(A)T_p(B)) = 4^p \int_{A \times B} dx dy (G_D(x, y))^p.$$

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